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Numerical Models and Observations of Water Motion in Green Bay, Lake Michigan

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NUMERICAL MODELS AND OBSERVATIONS OF WATER MOTION IN GREEN BAY, LAKE MICHIGAN

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CONTENTS

		PAGE
1.	Introduction	371
2.	Description of the numerical models	374
3.	Response of model 2 to a periodic variation of water level at its mouth	378
4.	Response of models 2 and 3 to a uniform wind field over the Bay	383
5.	Verification of model 2 by using field observations	391
6.	Concluding remarks	397
	References	398

Three numerical models are formulated for long-wave motion in a 180 km long gulf (Green Bay, Wisconsin) that opens into Lake Michigan. These models are used to investigate the response of the Bay to wind forcing and excitation by disturbances entering from the main Lake basin. Model simulations of water movements in the Bay have been done for two periods of 4 and 8 days respectively during 1969. Observed fluctuations in water level during these periods have been compared with the corresponding variations predicted by the models. Agreements and disagreements are discussed. These illuminate properties of the Bay's motion and raise some further questions.

1. Introduction

The gulf known as Green Bay, Wisconsin, here referred to as the Bay, is approximately 180 km long and opens into Lake Michigan (here referred to as the Lake) through several inter-island channels constituting the 'mouth'. These geographical features are illustrated in figure 1. Of special interest for the investigation described in this paper, Mortimer (1965) drew attention to the remarkably large oscillations of water level observed at the head of the Bay. His work was based on data given by continuous chart records of water level, covering a decade or so, obtained from recorders operated by the U.S. Army Corps of Engineers (Lake Survey) at coastal stations around the Lake. The records were compared and subjected to spectral analysis. In particular, the record from Green Bay City at the head of the Bay showed that amplitudes of oscillation of order 20–30 cm occur there, compared with say 2–3 cm at the mouth of the Bay and in the Lake generally. The magnification from the mouth to the head, thus made evident, was explained by Mortimer in terms of a double resonance between the first free longitudinal mode of oscillation of the Bay (of estimated period 10.8 h) and (i) the M_2 semidiurnal tide (12.4 h) and (ii, when excited) the first free longitudinal mode of oscillation of the Lake (9.0 h) acting at the Bay mouth. The properties of these barotropic modes of oscillation have been studied in detail by Mortimer & Fee (1976), who examined power spectra of observed water levels, and by Rao *et al.* (1976), who used a two-dimensional numerical method.

To provide deeper understanding of the motion in the Bay, the investigations into modes of oscillation and resonances described above need to be supplemented by theoretical studies that attempt to predict the water movements and then test those predictions against observed motions. Consistent with this requirement, the purpose of the present paper is to model the dynamics of the Bay, regarded as a gulf driven on the one hand by wind acting over its water surface and on the other hand by excitation at the mouth originating from water movements in the Lake. Interest centres on determining variations in water level and current caused by these applied forces. The main questions are the following. Can the extraordinary oscillations that are observed in the Bay be reproduced by the model? Does any one driving mechanism predominate? Under what circumstances do the oscillations attain their maximum amplitudes?

Three time-dependent numerical models are used to investigate separately the effects of mouth forcing and wind action. Then the latter two influences are combined in model simulations of observed water level extending over several days. Some good agreements are obtained between the model and the observations. The discrepancies encountered lead to speculations concerning the estimation of wind stress and the nature of the interaction between Bay and Lake.

The models are conventional in design, being based on the two-dimensional vertically integrated hydrodynamic equations frequently used in tidal and storm-surge computations. A well tested finite difference scheme is employed in deriving time-stepping numerical solutions. The calculation grids are plane; each has a uniform square mesh of either 2 or 4 km side length.

An earlier numerical model by Birchfield & Murty (1974) investigated the wind-driven circulation in lakes Michigan and Huron, including Green Bay. The main feature of this work was the use of a stretched coordinate system by which the irregular boundary of the lakes was mapped onto the boundary of a series of adjoining rectangles, the dynamical equations then being solved over a square mesh covering the rectangles. Results from that model demonstrated that, in circumstances under which the first longitudinal mode of Lake Michigan was excited, there was a large magnification in amplitude from the mouth to the head of Green Bay. In comparison, the present work focuses more attention on processes and variations *within* the Bay and is concerned to a greater degree with the comparisons between models and observed water level changes.

By definition, Green Bay is a part of Lake Michigan (figure 1). However, for convenience in this paper, we shall generally think of the Lake as excluding the Bay so that a clear distinction can be drawn between the relatively shallow Bay area and the deeper Lake basin with which it communicates.

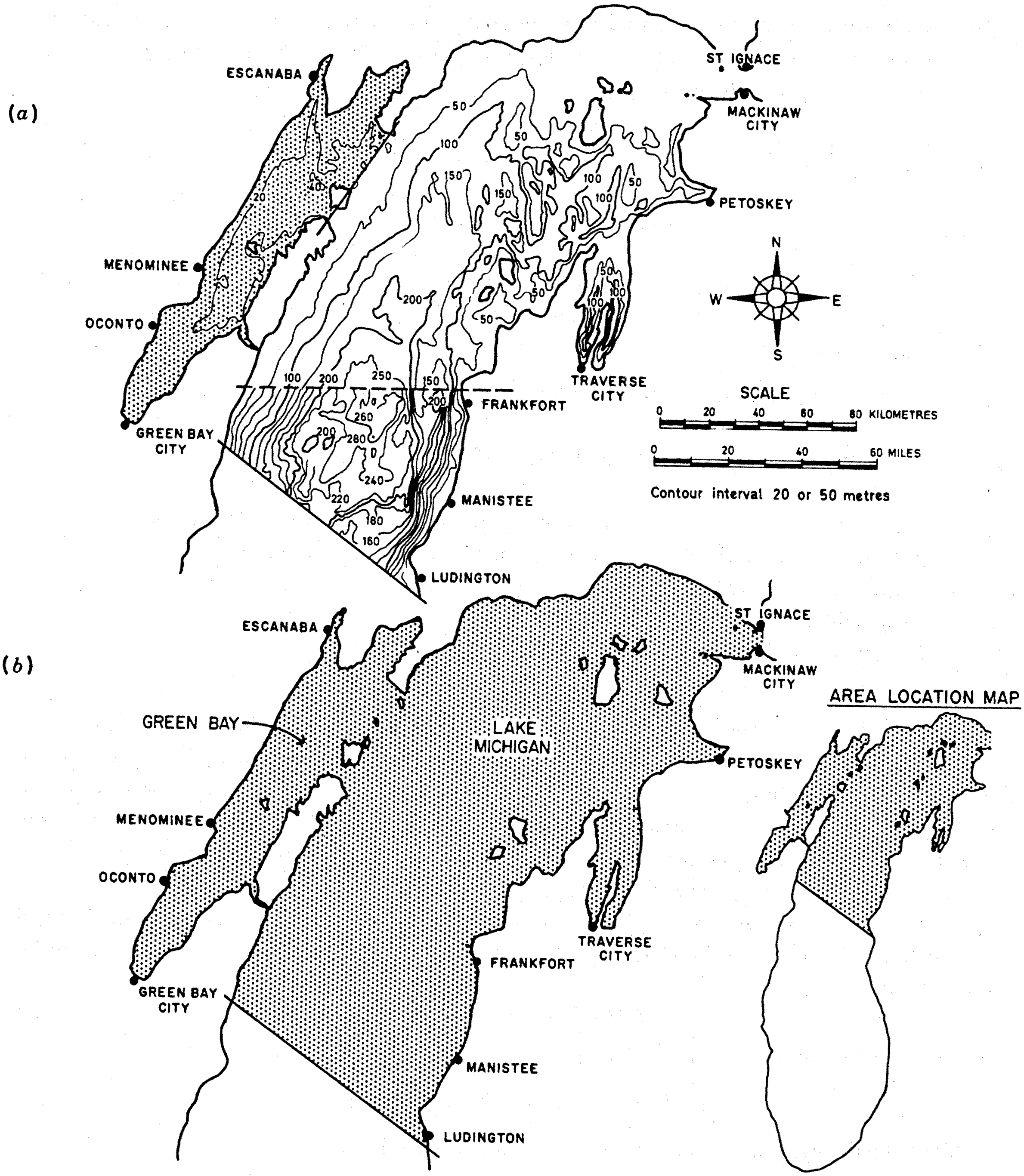


FIGURE 1. Charts of the northern part of Lake Michigan, including Green Bay. Shaded regions indicate model areas. Thus (*a*) shows the area covered by models 1 and 2 of the Bay alone and (*b*) the area covered by model 3 of the Bay and northern Lake Michigan.

2. Description of the numerical models

The vertically integrated differential equations of continuity and motion, commonly employed in tidal and storm-surge computations (see, for example, Flather & Heaps 1975), are used. These may be written in the form

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\{(h+\zeta)\,u\} - \frac{\partial}{\partial y}\{(h+\zeta)\,v\}, \tag{1}$$

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \zeta}{\partial x} - \frac{ku(u^2+v^2)^{\frac{1}{2}}}{h+\zeta} + \frac{F_s}{\rho(h+\zeta)}, \tag{2}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial \zeta}{\partial y} - \frac{kv(u^2+v^2)^{\frac{1}{2}}}{h+\zeta} + \frac{G_s}{\rho(h+\zeta)}, \tag{3}$$

where the notation is:

x, y	Cartesian coordinates in the horizontal plane of the undisturbed water surface,
t	time,
ζ	elevation of the water surface above the undisturbed level,
h	undisturbed depth of water,
u, v	components of depth–mean current in the directions of increasing x, y respectively,
F_s, G_s	components of wind stress on the water surface in the x, y directions,
f	the Coriolis parameter,
k	a coefficient of bottom friction,
ρ	the density of the water,
g	the acceleration due to the Earth's gravity.

The total depth of water at any instant is $h+\zeta$ where ζ is a time-varying part defining water level variations. Basic assumptions employed in the derivation of equations (1)–(3) involve the neglect of friction in horizontal planes, the omission of the convective accelerations and the use of the hydrostatic law. The water is considered as being homogeneous, the direct effect on the Bay of the tide-generating forces is ignored, and the quadratic law of bottom stress is postulated. Parameters f and k are regarded as constants.

The equations are solved for ζ, u, v through time over the Bay alone and also over the Bay and the northern part of the Lake (figure 1). The motion determined is generated from an initial state of rest,

$$\zeta = u = v = 0 \quad \text{at} \quad t = 0, \tag{4}$$

under the action of prescribed wind stress

$$F_s = F_s(x, y, t), \quad G_s = G_s(x, y, t) \tag{5}$$

and prescribed elevation ζ along the open sections of boundary

$$\zeta = \hat{\zeta}(x, y, t). \tag{6}$$

Along a closed land boundary the normal component of current q_n is set to zero:

$$q_n = u\cos\alpha + v\sin\alpha = 0, \tag{7}$$

where α denotes the inclination of the normal to the x axis.

As described in detail by Flather & Heaps (1975), solutions are derived numerically by a finite-difference approach, with values of ζ, u, v evaluated at discrete points on a uniform rectangular grid. The x and y coordinate axes are parallel to the grid lines. Based on equations (1)–(3) written in finite-difference form, a forward time-stepping technique yields the horizontal

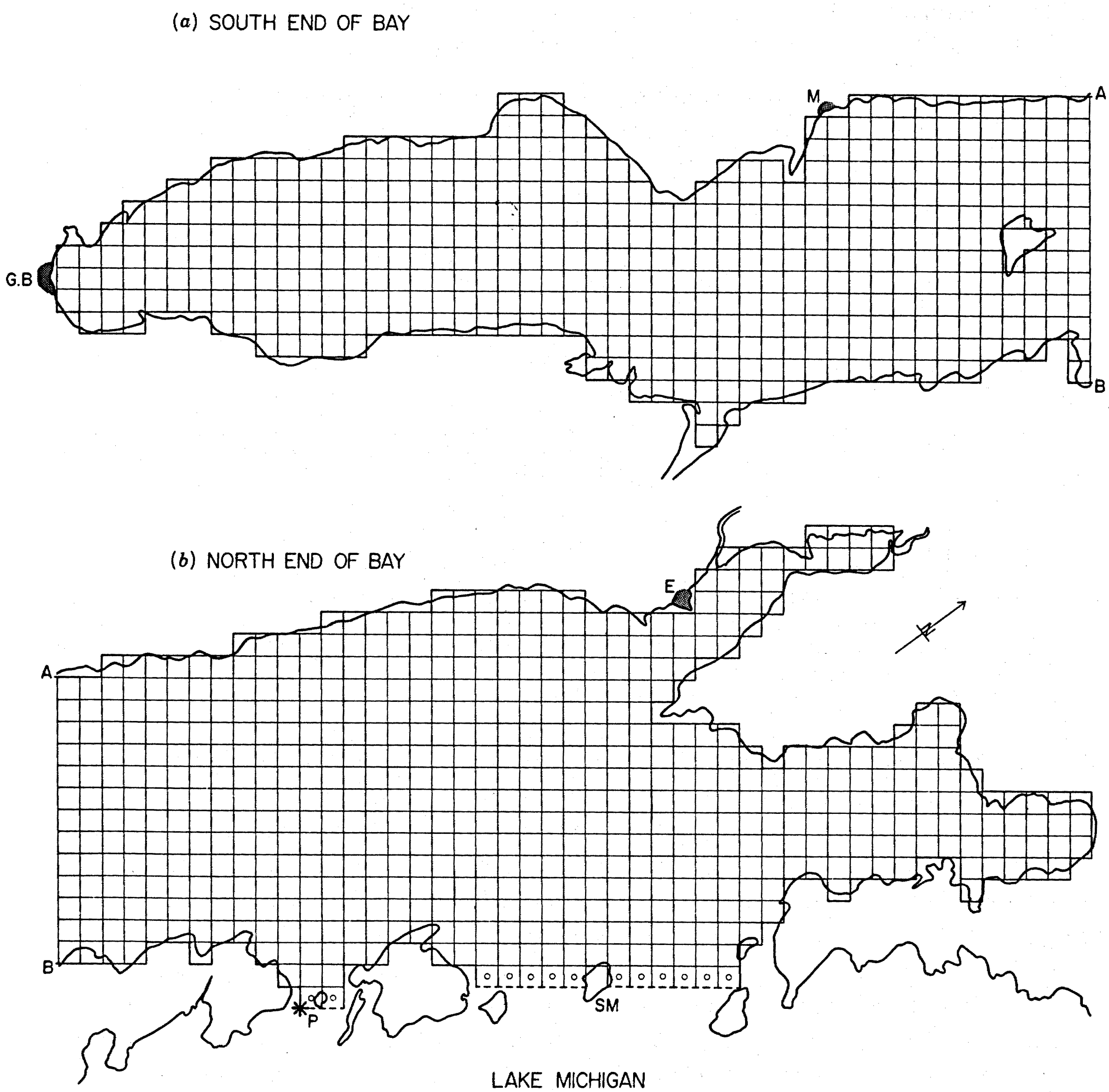


FIGURE 2. Grid network for model 1 of Green Bay, showing (*a*) the south end of the Bay and (*b*) the north end of the Bay. A 2 km square mesh is employed. Broken lines mark out the open boundary (the mouth) and circles denote elevation input points adjacent to that boundary. Key: GB, Green Bay City; M, Menominee; E, Escanaba; P, Plum Island; SM, St Martin Island; AB, common section between the southern and northern areas of the model. Asterisk near P shows position of current meter referred to in figure 13.

fields of ζ, u, v (as defined by the grid values) at intervals Δt through advancing time. This procedure is here applied to the computation of long-wave motion in the Bay with three models summarized as follows.

Model 1, delineated in figure 2, covers the entirety of the Bay and is based on a horizontal grid network with a square mesh of side 2 km. The open boundary of this model lies across the mouth of the Bay, traversing the island region that separates the Bay from the Lake. Other edges of the network, as shown, represent closed coastal boundaries.

Model 2, delineated in figure 3, covers essentially the same area as model 1 but has a coarser grid with a square mesh of side 4 km. The open boundary is close to that of model 1. The grid lines of model 2 coincide with alternate grid lines of model 1.

Model 3, delineated in figure 4, consists of model 2 extended into the northern part of the Lake. This larger model, with a 4 km grid, has an open boundary coinciding with a central section of the Lake and another lying across the Straits of Mackinac.

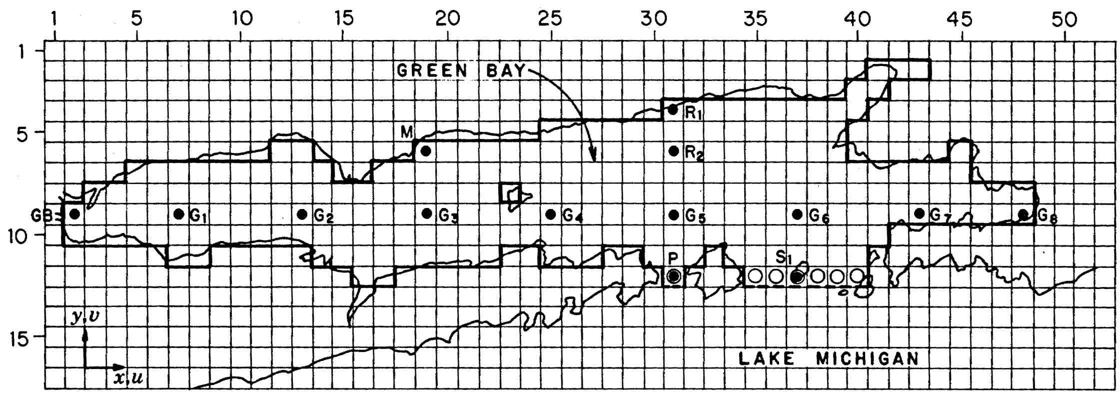


FIGURE 3. Grid network for model 2 of Green Bay, with a 4 km square mesh. The x, y coordinate directions are shown and the rectangular numbering system for the central points of the square elements is indicated. Key: ——, actual coastline; ——, model coastal boundary; - - -, model open boundary; ○, open-boundary point for elevation input; ●, point for which computed elevations are plotted against time, including GB Green Bay City, M Menominee, P Plum Island, S_1 St Martin Island.

In each model, bottom topography is defined in terms of chart depths (defining h) evaluated at the centres of the grid squares while coastline is approximated (as shown) by a step-like boundary marked out on the grid lines. In § 5 the chart depths used are modified to take account of variations in mean water level. Clearly, the finer the grid, the better the representation of depth distribution and coastal configuration. Surface elevation ζ is evaluated at the centre of each grid square; open-boundary elevation values ξ (equation (6)) are prescribed at ζ points marked by circles in the figures. Current component u is evaluated at the midpoints of the y-directed sides of each square element and current component v at the midpoints of the x-directed sides. Then, setting either $u = 0$ or $v = 0$ on a model boundary designated as closed fulfils the condition of zero normal flow there (equation (7)). Current components are obtained at the centre of each element by averaging their values on the opposite sides. In prescribing wind stress (equation (5)), F_s is evaluated at each u point and G_s at each v point. These values are for insertion in the finite-difference equivalents of equations (2) and (3). As illustrated in figures 3 and 4, a rectangular numbering system locates the centre of each grid square. Thus, for example, G_6 in figure 3 is referred to as the point (9, 37).

Computations with the models were done on the UNIVAC 1108 computer at the University of Wisconsin mainly during 1969. Additional runs were done during 1970–72.

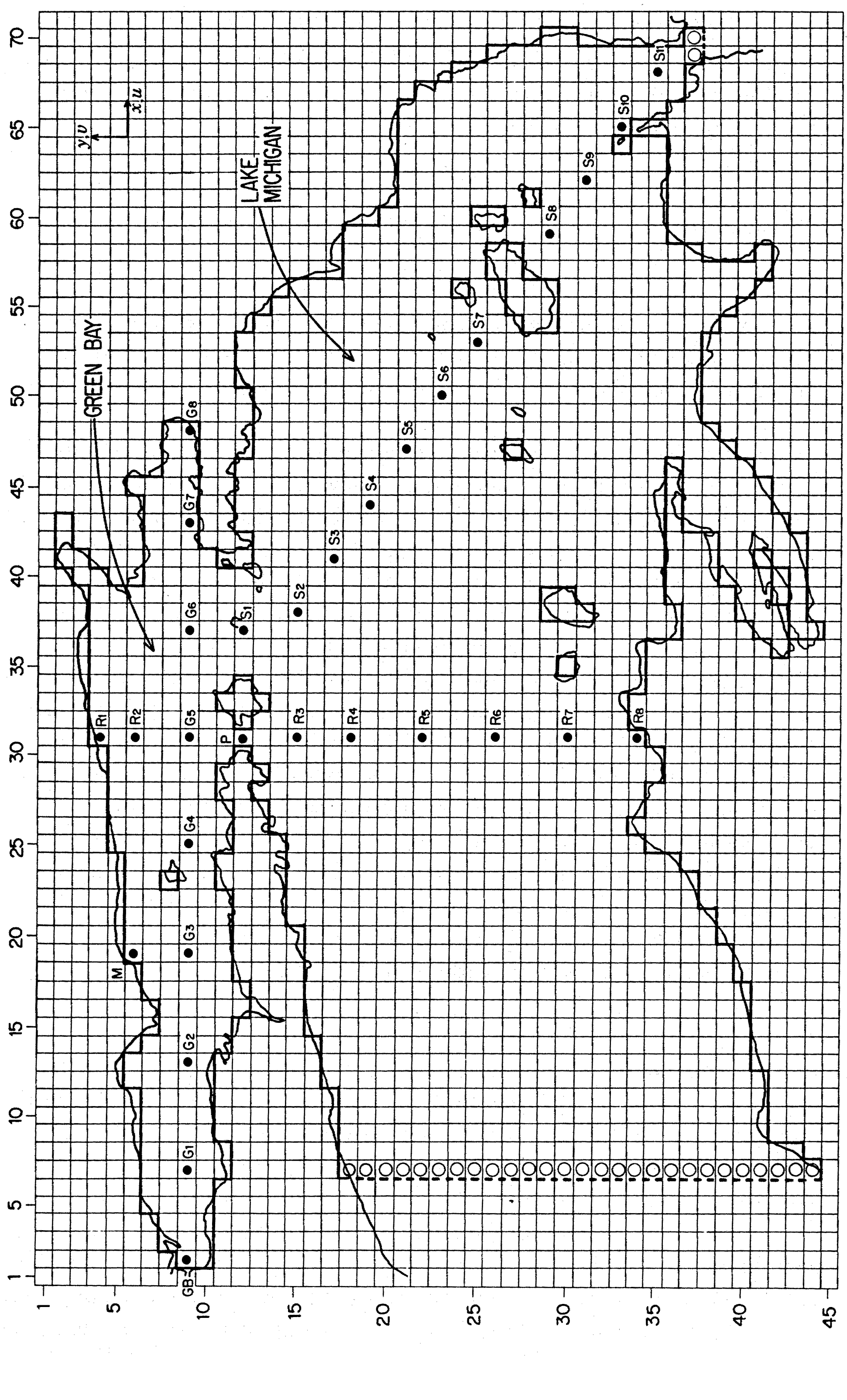


FIGURE 4. Grid network for model 3 of Green Bay and northern Lake Michigan, with a 4 km square mesh. The x, y coordinate directions are shown and the rectangular numbering system for the central points of the square elements is indicated. Key as for figure 3.

3. Response of model 2 to a periodic variation of water level at its mouth

In a first set of numerical experiments, considering the Bay alone, models 1 and 2 were run with zero wind stress over the water surface,

$$F_s = G_s = 0, \tag{8}$$

and a simple harmonic oscillation of water level applied uniformly across the mouth,

$$\zeta = A \sin (2\pi t/T), \tag{9}$$

where A is the amplitude and T the period of oscillation. The motion generated from an initial state of rest in response to this mouth excitation was computed through time until, following the frictional damping of transients, a periodic co-oscillation had been established at all points within the Bay.

With use of model 2, the co-oscillation was determined with $A = 2$ cm for periods $T = 12.4$, 10.8, 10.0, 9.8, 9.6, 9.2, 9.0, 8.8, 8.0 h. An amplitude A of 2 cm is typical of oscillations observed at the Bay's mouth; the range of T includes the periods of the fundamental natural surface modes of the Lake and the Bay, taken as 9.0 and 10.8 h respectively (Mortimer 1965), and the period of the semidiurnal M_2 tide, 12.4 h. An acceptable time step Δt for the computations was 150 s: in fact, a nearby value commensurate with period T was chosen in each case to facilitate the regular output of ζ, u, v throughout a cycle. A closely periodic response was obtained after passing from rest through 21 cycles of forcing. However with $T = 12.4$ h it was found necessary to go to 33 cycles and with $T = 10.8$ h to 26 cycles, owing presumably to a lower rate of frictional damping in the oscillations of lower frequency. Originally the intention was to use model 1 for all these computations. Unfortunately it became clear at an early stage that, with $\Delta t = 75$ s, the running of model 1 was going to be too costly for completing the full programme of work. Model 2 was subsequently formulated and put to the task. Because of its coarser network, that model has to be regarded as less accurate than model 1. On the evidence gained from these computations, it appears that differences of up to 15 % can arise between corresponding results from the two models. However, such discrepancies are scarcely important in the context of the present paper. In the runs with model 2, and later with model 3, we took

$$f = 1.0355 \times 10^{-4}\,\mathrm{s}^{-1}, \quad k = 0.0026. \tag{10}$$

Derived from the computations, a frequency response curve relating to surface level at GB (9, 2), a point at the head of the Bay adjacent to Green Bay City, is given in figure 5. The magnification in amplitude, M, and the phase lag, θ, of the surface oscillation at GB relative to the surface oscillation applied at the mouth are plotted against T for $8\,\mathrm{h} \leqslant T \leqslant 12.4$ h. Manifestly the resonant peak is at 9.75 h and not at 10.8 h, the period estimated by Mortimer (1965) for the first free longitudinal surface mode of the Bay. As explained by Heaps (1975), such a result is to be expected since the peak, here calculated, occurs at the natural period of the Bay with a node *imposed* at the mouth and not at the natural period of the Bay communicating freely with the Lake. Figure 5 shows that Lake mode 1, the first free longitudinal surface mode of the Lake, with a 9.0 h period, is magnified in amplitude by a factor of 9.8 from the mouth to the head of the Bay, the magnification being accompanied by a phase increase of 166°. Correspondingly, the M_2 tide of the Lake, with a 12.4 h period, is magnified in amplitude along the length of the

Bay by a factor of 3.5, this being accompanied by a phase increase of 5°. Table 1 compares these magnifications and phase lags with corresponding magnifications and phase lags, obtained from Mortimer (1965) and Mortimer & Fee (1976), for the Bay mouth on Mackinaw City and Green Bay City on Mackinaw City. The locations of Mackinaw City, MC, and Green Bay City, GB, are shown in figure 1. To a satisfactory order of approximation table 1 confirms that for magnification

$$(\text{GB/Bay mouth}) \times (\text{Bay mouth/MC}) = (\text{GB/MC})$$

and for phase lag

$$(\text{GB/Bay mouth}) + (\text{Bay mouth/MC}) = (\text{GB/MC}),$$

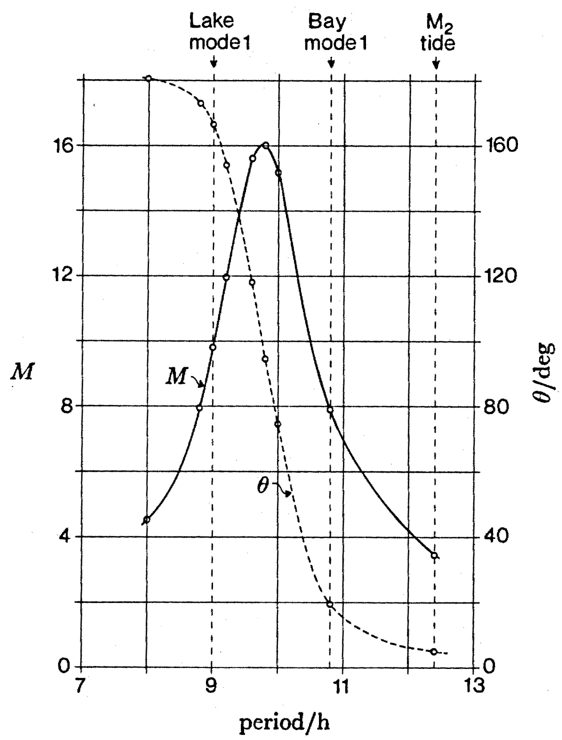
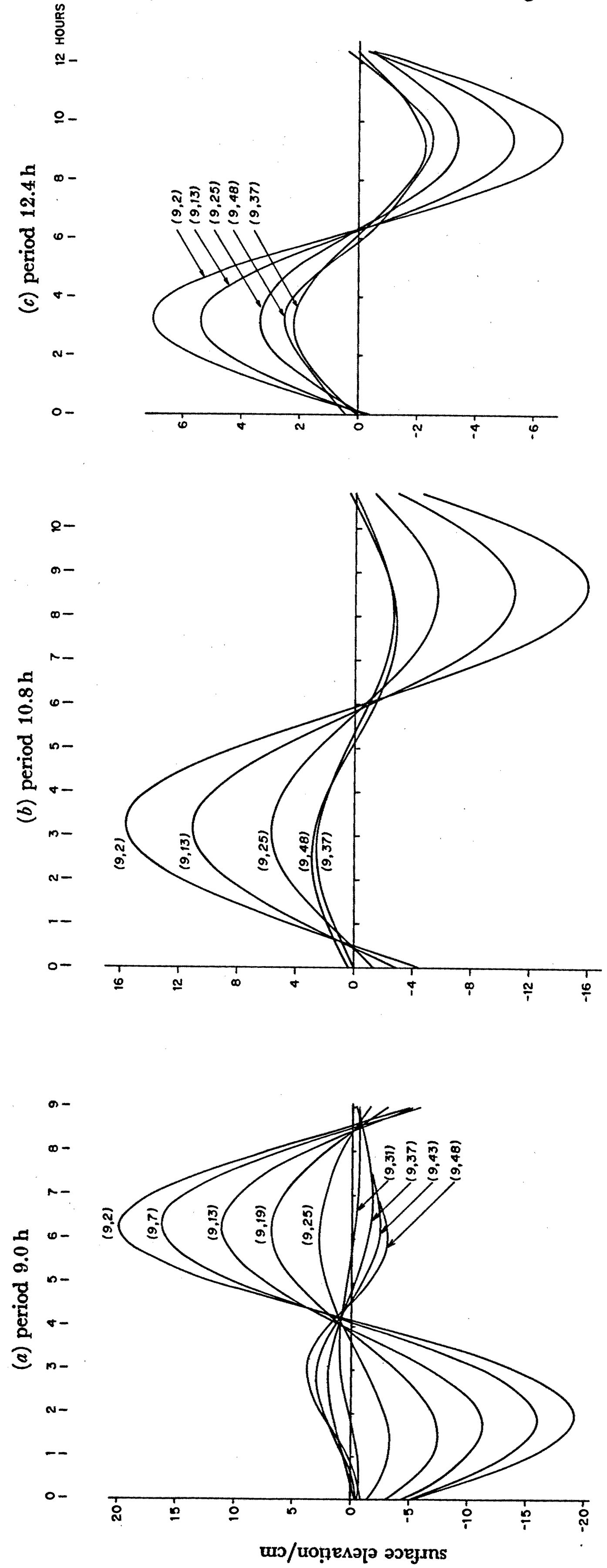


Figure 5. Periodic response of the water level at GB (9, 2) (Green Bay City) to a simple harmonic oscillation of surface elevation of amplitude 2 cm applied across the mouth of the Bay at points (12, 31), (12, 35) to (12, 40). The magnification M and the phase lag θ of the oscillation at GB relative to that at the mouth are plotted against the period of the applied oscillation. Results from model 2. See figure 3 for point locations.

at least for Lake mode 1. However, for the M_2 tide, the magnification of the Bay mouth on MC is not available and the sum of the phase lags of GB on the Bay mouth and the Bay mouth on MC falls short by 14° of the observed phase lag, 35°, of GB on MC. The reason for this discrepancy is presently unknown but the tides in the Lake certainly appear to require more detailed attention (Mortimer & Fee 1976) and until more work is done on them the phase lag, 16°, of the Bay mouth on MC has to be regarded as tentative. The 5° phase lag of GB on the Bay mouth agrees with an unpublished result obtained by N.S.H. from a one-dimensional tidal model of the Bay designed along the lines described by Proudman (1953, pp. 325–329). Nevertheless, future work may show that increasing the coefficient of friction in model 2 might reduce the 14° deficiency without significantly damping amplitudes of oscillation in the Bay.

We now present further results of the computations of periodic motion with model 2. Time variations of surface elevation and mouth current for Lake mode 1 (period 9.0 h), for Bay mode 1


(a) period 9.0 h
(b) period 10.8 h
(c) period 12.4 h
surface elevation/cm
12 HOURS
(9,2)
(9,7)
(9,13)
(9,19)
(9,25)
(9,31)
(9,37)
(9,43)
(9,48)

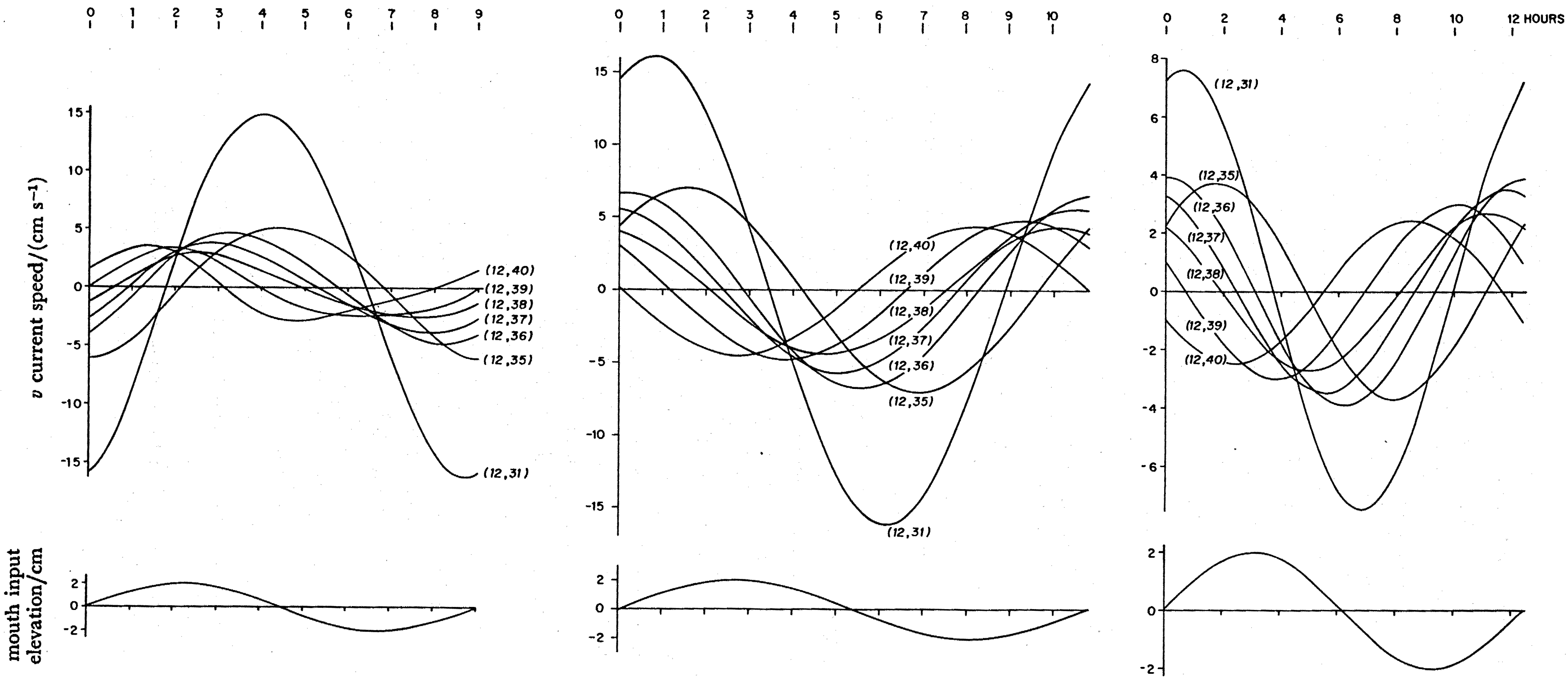


FIGURE 6. Periodic response of Green Bay to a simple harmonic oscillation of surface elevation, applied across the mouth, of amplitude 2 cm and period (*a*) 9.0 h, (*b*) 10.8 h and (*c*) 12.4 h. Derived from computations with model 2. The curves show time variations of elevation at various points (9, 2) to (9, 48) along the longitudinal axis. Also time variations of v current directed normally across the mouth at points (12, 31), (12, 35) to (12, 40). All these points are marked in figure 3.

(period 10.8 h), and for the M_2 tide (period 12.4 h) are shown in figure 6*a–c* respectively. In figure 6*a*, plots of elevation against time are given for nine points along a central axis of the Bay: GB (9, 2), G_1 (9, 7), G_2 (9, 13), G_3 (9, 19), G_4 (9, 25), G_5 (9, 31), G_6 (9, 37), G_7 (9, 43), G_8 (9, 48). These points are indicated in figure 3. Plots of v current, directed into the Bay and acting normally across its mouth, are given for the six model points in the St Martin Island passage (12, 40)–(12, 35) and for the single point in the Plum Island passage (12, 31).

Table 1. Amplitude magnifications and phase lags for (i) the first free longitudinal surface mode of Lake Michigan (Lake mode 1) and (ii) the M_2 tide

(GB, Green Bay City; MC, Mackinaw City.)

	amplitude magnification		phase lag/deg	
	Lake mode 1	M_2 tide	Lake mode 1	M_2 tide
GB/Bay mouth†	9.8	3.5	166	5
Bay mouth/MC‡	0.55	—	9	16
GB/MC‡	6	5	175	35

† From model 2 of the present paper.
‡ From Mortimer (1965) and Mortimer & Fee (1976).

Figure 6*a* shows that the elevation amplitude in the 9.0 h mode builds up steadily from G_5, through G_4–G_1, to a maximum at GB. Moving in the opposite direction, from G_5, through G_6, G_7, to G_8, there is only a small increase in amplitude. The oscillations on either side of G_5 are essentially out of phase, those nearer the mouth being approximately in phase with the forcing oscillation at the mouth. Thus, the oscillation at the head is almost out of phase with that at the mouth. At G_5 the amplitude is small and near a minimum. According to the computations, amplitude is nowhere permanently zero in the Bay. Evidently the point of minimum amplitude corresponds to the amphidromic point derived from the frictionless normal mode calculations of Rao *et al.* (1976, fig. 4). The inclusion of friction alters the phases of incident and reflected waves and removes the cancellation between them which produces the amphidrome. Figure 6*a* shows that the currents into the Bay occur with progressively later phase when proceeding from north to south across the mouth; in the north they tend to be in unison with the mouth input elevation. The currents to the south appear to feed the main oscillations of the southern end of the Bay; those to the north may perhaps be associated more with the smaller oscillations at the northern end. Manifestly the mouth currents are greatest in the Plum Island passage; but their phase in this passage does not quite fit into the above-mentioned phase progression across the mouth.

Figure 6*b* gives comparable plots of elevation and current for the 10.8 h mode. The figure shows that elevation along the whole length of the Bay is nearly in phase with that at the mouth, its amplitude building up towards GB at the head. There is no amplitude minimum of elevation, consistent with the results of the inviscid normal mode calculations of Rao *et al.* (1976, fig. 7) which predict no amphidromic point within the Bay but one lying just outside the Bay's mouth in the Lake. The currents directed into the Bay across the mouth again change in phase from north to south; they lead the elevations by 90–180°. The oscillatory response along the Bay now occupies less than one-quarter of a wavelength; in the 9.0 h mode it occupies more than one-quarter of a wavelength.

Figure 6*c* displays plots of elevation and current in the 12.4 h tidal mode. Clearly, the response of the Bay in this mode is very similar to that in the 10.8 h one: spectrally, both modes lie on the same side of the resonant peak in figure 5. However, in accordance with the values plotted in figure 5, the magnification of elevation amplitude along the Bay is less for the tides than for the Bay mode. The phase lag θ from the mouth to the head is also smaller for the tides.

4. Response of models 2 and 3 to a uniform wind field over the Bay

In a further experiment with model 2, computations were done to determine the motion in the Bay produced by a uniform wind field of 8 m s^{-1} from N 38°E, suddenly created over the whole Bay at time $t = 0$ and then held constant. That wind blows along the main axis of the Bay towards the head at Green Bay City, acting in the negative x direction, see figure 3. Wind stress τ was calculated from the formula (after Heaps 1965)

$$\tau = 1.25\,cw^2, \tag{11}$$

where τ is in newtons per square metre, w is the wind speed in metres per second and c is a drag coefficient:

$$\left.\begin{aligned} 10^3c &= 0.554 && 0 \leqslant w \leqslant 5 \\ &= -0.12+0.137w && 5 < w < 19 \\ &= 2.513 && w \geqslant 19. \end{aligned}\right\} \tag{12}$$

Accordingly, model 2 was run from a state of rest at $t = 0$ with a wind stress over the water surface given by

$$\left.\begin{aligned} F_s &= -1.25\times(-0.12+0.137\times 8)\times 10^{-3}\times 8^2 \\ &= -0.078\,\mathrm{N\,m^{-2}} \\ G_s &= 0. \end{aligned}\right\} \tag{13}$$

At the same time, surface elevation across the mouth was set permanently to zero:

$$\zeta = 0. \tag{14}$$

With these inputs, model 2 produced the results illustrated by the continuous lines in figure 7. Time plots of surface elevation are shown for Green Bay City, GB (9, 2), and for Menominee, M (6, 19), covering a period of 40 h following the onset of the forcing at $t = 0$. Clearly, the wind generates damped oscillatory motion in the Bay. A main oscillation may be recognized with a period of roughly 10 h. That oscillation may be identified with the free longitudinal mode (period 9.75 h), corresponding to the resonant peak in figure 5 and based on the presence of a node (zero elevation change) at the Bay's mouth.

The assumption of a node at the mouth, embodied in equation (14), is admittedly unreal; and because of the Earth's rotation it gives rise to fictitious local flow gyres in the mouth region. The mouth condition should be modelled so that the Bay communicates with the Lake in a manner more closely resembling the situation in Nature. With this in mind, motion generated by the same wind stress field over the Bay as before (equation (13)) was computed with use of model 3. This model, shown in figure 4, covers the Bay and also the northern part of the Lake, and ensures, within the limits of its resolution, a continuous hydrodynamic connection between the Bay and the Lake. Its open boundaries, across a central section of the Lake and across the Straits of Mackinac, are well removed from the Bay mouth. The nodal condition (14) is imposed along those boundaries, and it is assumed that setting open-boundary elevations to zero in this way does not significantly influence the computed wind-generated motion in the Bay itself. To the extent that water movements generated within the Bay do not drive whole-lake motions to any appreciable extent, the assumption would appear to be reasonable. Work for the future (not attempted here because of computer limitations) is to model the Bay with the whole of the Lake with use of the present 4 km grid. This would avoid having a long open boundary across the Lake.

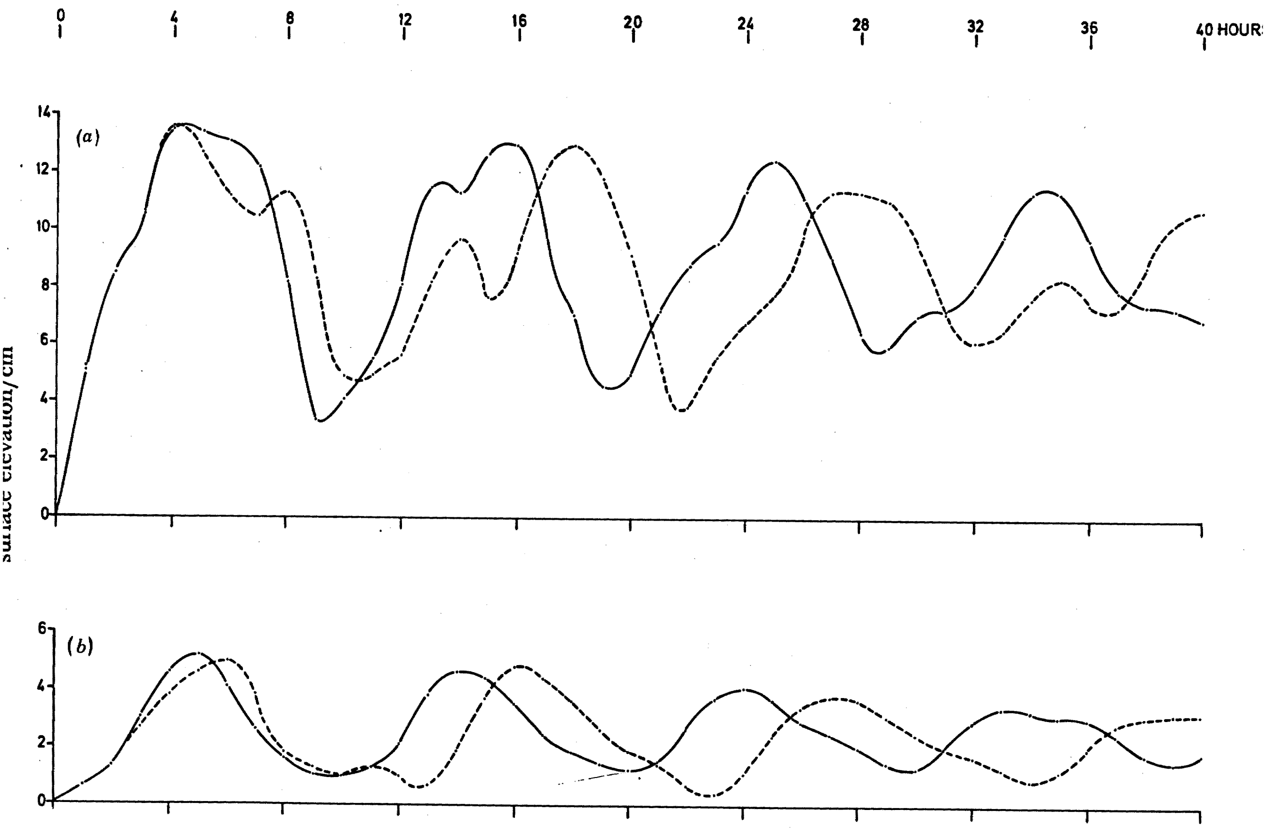


FIGURE 7. Water level responses at (*a*) Green Bay City, point (9, 2), and (*b*) Menominee, point (6, 19), to a uniformly distributed wind of 8 m s^{-1} from N 38° E suddenly created at time $t = 0$ over Green Bay and subsequently maintained constant. Surface elevation is plotted against time at each point. The wind is directed along the length of the Bay from the mouth to the head. Key: ——, results from model 2 (Green Bay alone, surface elevations set permanently to zero across the mouth; see figure 3); – – –, results from model 3 (Green Bay combined with northern Lake Michigan, surface elevations set permanently to zero across open boundaries in the Lake; see figure 4).

The elevations at GB and M, computed with model 3 and resulting from the wind stress field (13) over the Bay, are shown by the dashed line in figure 7. The corresponding output from model 2 is also shown in figure 7, and it is apparent that, while amplitudes of the main damped oscillatory motion are determined similarly by the two models, there is a phase change of 180°, over the first 36 h, between corresponding oscillations derived from them. Therefore, letting T_1 h be the periodicity with model 3, and taking 9.75 h to be the periodicity with model 2 as inferred above, we have, on equating rates of phase change,

$$\frac{360}{9.75\,\mathrm{h}} - \frac{360}{T_1} = \frac{180}{36\,\mathrm{h}},$$

giving

$$T_1 = 11.2\,\mathrm{h}.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This result confirms that the oscillations in the Bay computed with model 3 have a period consistent with that of Bay mode 1, estimated from observations to be 10.8 h (Mortimer 1965).

Figures 8–12, 14 and 15 display further results obtained from model 3, describing the response of the Bay and Lake to the wind stress field given by equation (13), suddenly created over the

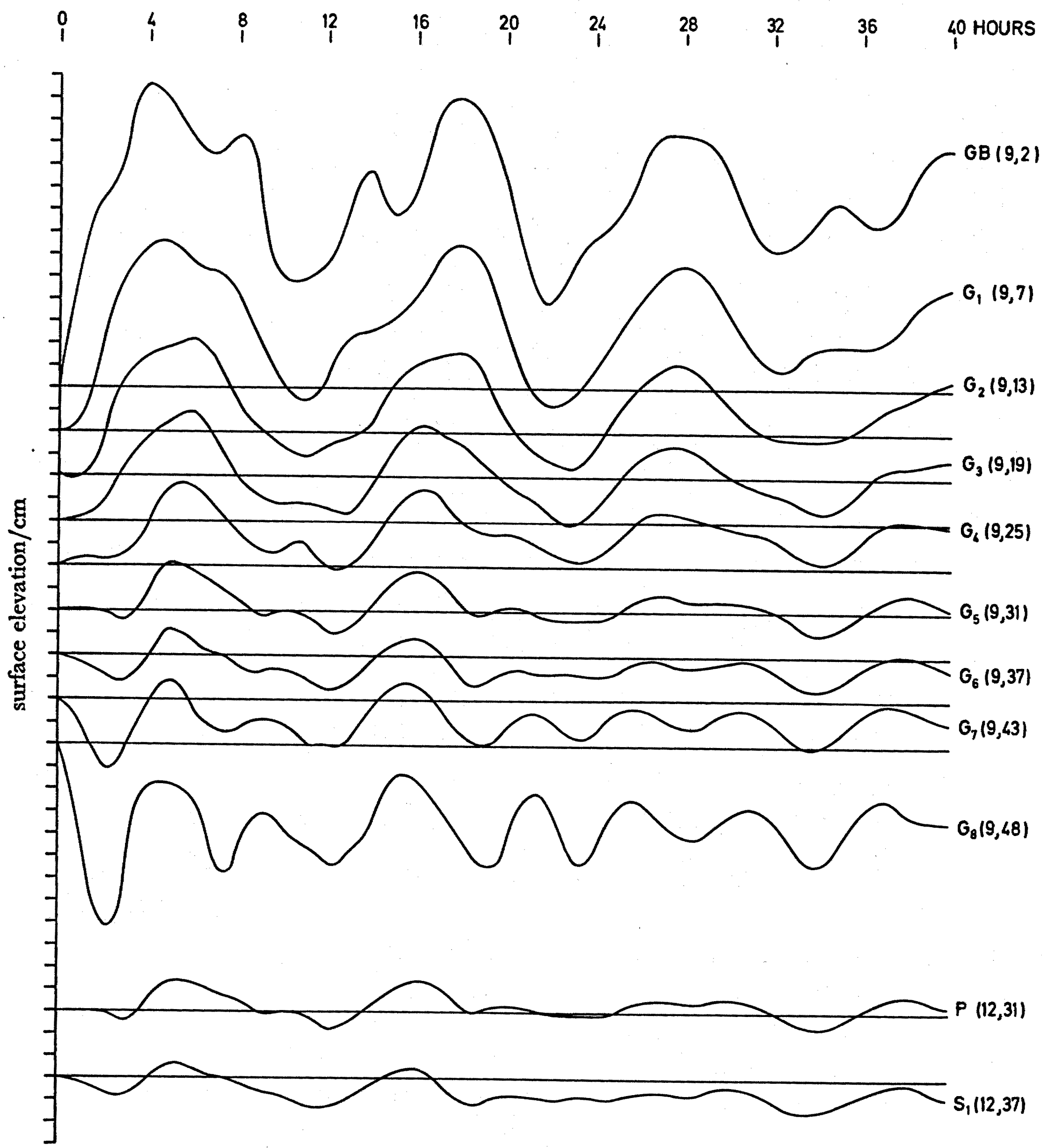


FIGURE 8. Water level variations in Green Bay due to a uniformly distributed wind of 8 m s^{-1} from N 38°E suddenly created over the Bay at t = 0 and then kept constant (wind field W_L). Time plots of surface elevation are shown in order corresponding to points GB, G_1–G_8 along the axis of the Bay and points P, S_1 at the mouth (see figure 4). Results from model 3.

Bay at time $t = 0$, associated with a uniform wind field W_L of 8 m s^{-1} directed along the length of the Bay towards the head. In figure 8, time plots of surface elevation are shown for points GB, G_1–G_8 along the Bay's axis and for two points at the Bay's mouth: P near Plum Island and S_1 near St Martin Island. These positions are marked in figure 4. Excitation of the first Bay mode, period 10.8 h, is clearly evident from the plots. The magnification in amplitude, from the mouth to the head of the Bay, is roughly comparable with that predicted by figure 5. The second Bay mode with a period of approximately 5 h is also in evidence, particularly at each end of the Bay.

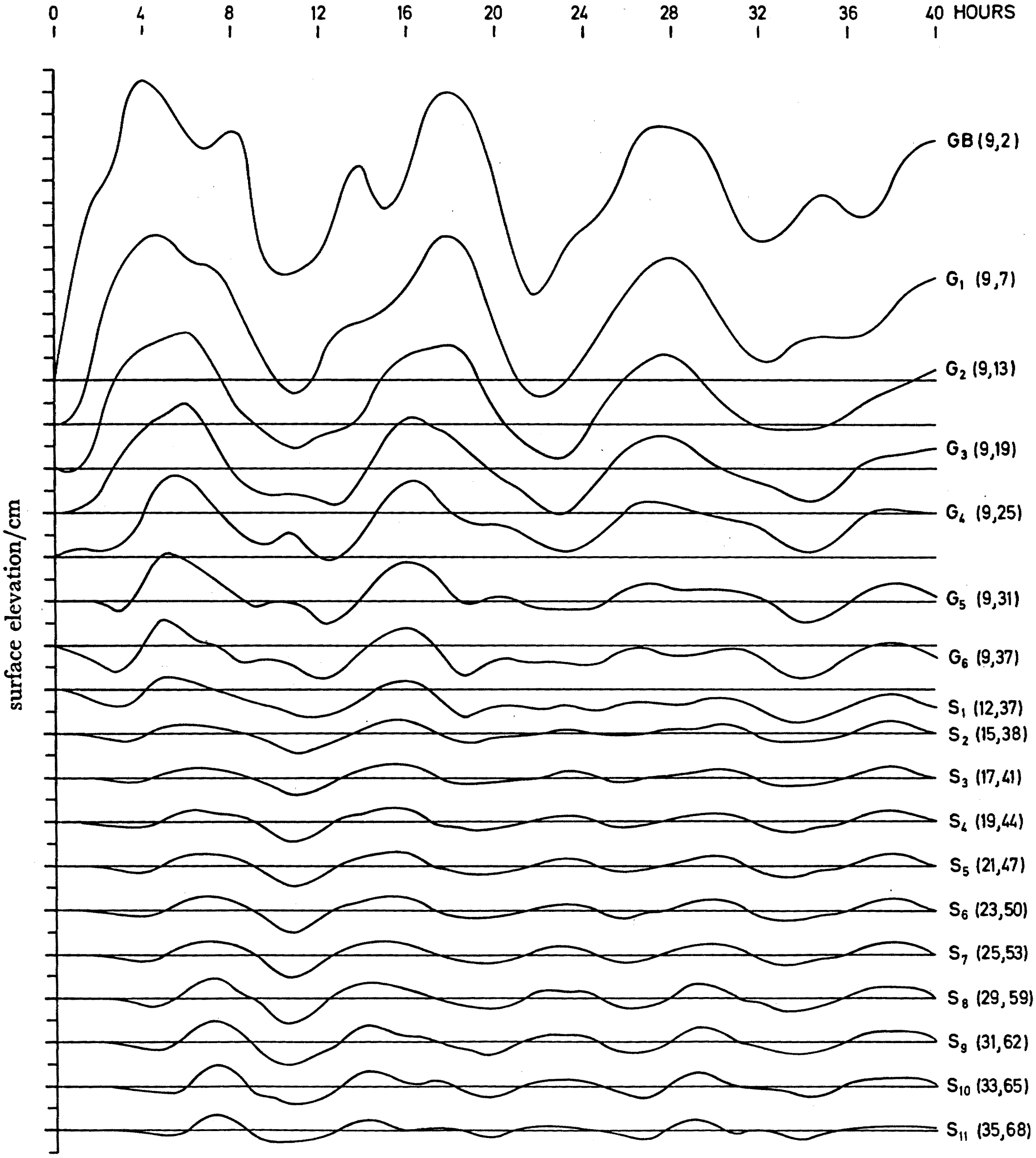


FIGURE 9. Water level variations in Green Bay and Lake Michigan due to wind field W_L. Time plots of surface elevation are shown in order corresponding to points GB, G_1–G_6 along the axis of the Bay and then points S_1–S_{11} along a line in the Lake from the mouth of the Bay to the Straits of Mackinac (see figure 4). Results from model 3.

However, at G_2 and G_3 it can scarcely be seen, a result consistent with the position of an amphidromic point in that region as computed for the second Bay mode by Rao *et al.* (1976, fig. 7). Spectra of water level fluctuations at Green Bay City, Escanaba and Plum Island (locations marked in figure 2), presented by those authors, disclose signals that correspond in period to both the first and the second Bay modes recognized here.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Figure 9 presents time plots of elevation at GB, G_1–G_6 along the axis of the Bay followed by similar plots at S_1–S_{11} along a line stretching diagonally across the northern part of the Lake from the Bay mouth to the Straits of Mackinac (figure 4). The plots indicate that an initial depression and rise in water level at the mouth spreads into and across the Lake. Subsequently, the Lake levels oscillate coherently with apparent periods of between 7 and 9 h. Amplitudes are small but relatively undamped; they tend to increase towards the Straits of Mackinac, reflecting a property of the second Bay mode determined by Rao *et al.* (1976, fig. 7), who predicted amphidromes in the Lake just outside the Bay mouth for both the first and second Bay modes. These amphidromic influences might perhaps be detected in the near-zero elevations at S_2.

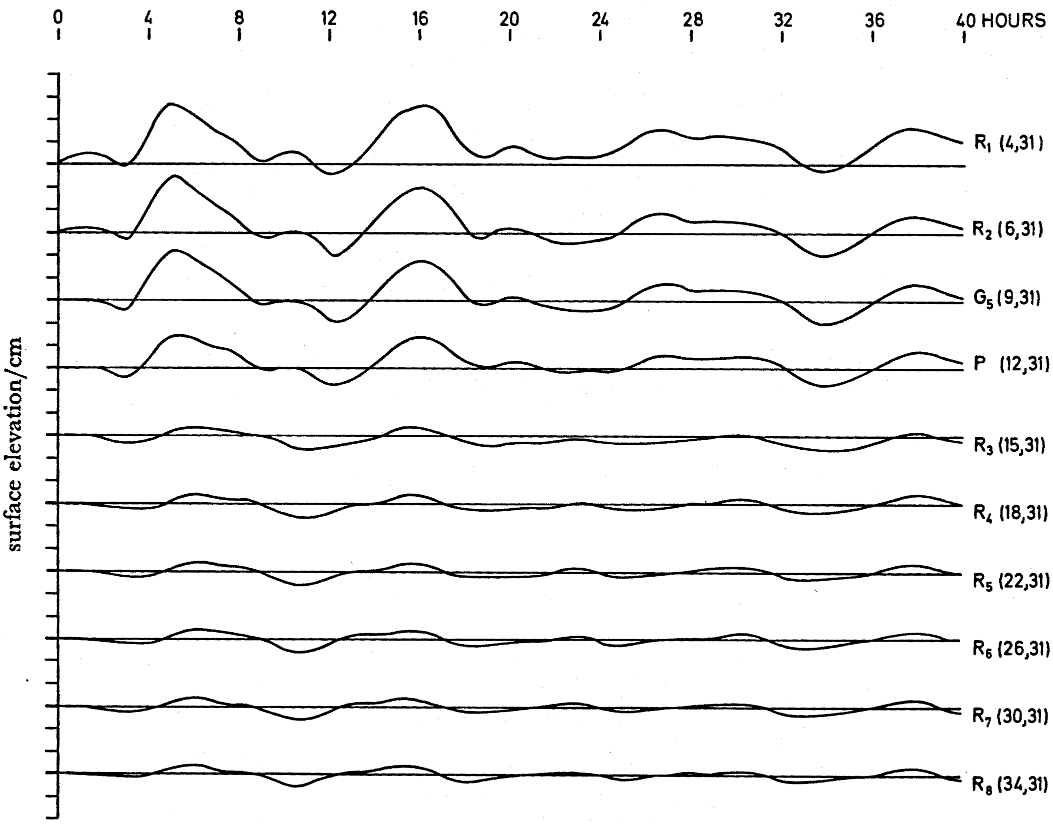


FIGURE 10. Water level variations in Green Bay and Lake Michigan due to wind field W_L. Time plots of surface elevation are shown in order corresponding to points R_1, R_2, G_5, P, R_3–R_7, R_8 on a line crossing the Bay and Lake transversely (see figure 4). Results from model 3.

Figure 10 shows time plots of elevation at R_1, R_2, G_5, P, R_3–R_8 along a line crossing the Bay and Lake transversely (see figure 4). Again the oscillatory motion generated by wind in the Bay is seen to spread out into the Lake, with a noticeable diminution in amplitude. Thus the Bay drives the Lake, but only into a state of relatively small oscillation. That oscillation appears to have a strong spatial coherence, and its own characteristic mode different to that of the Bay. The Bay signal is recognizable in spectra from several Lake stations. (See G in fig. 11 of Mortimer & Fee (1976).)

Figure 11 complements the time variations in figures 8–10 by showing the distribution of surface elevation over the Bay at a time 40 h after the onset of the wind. Contours demonstrate that the wind maintains a gradient of elevation along the length of the Bay, raising water levels towards the head and lowering them in the two northern gulfs. The influence of the Lake keeps elevations near to zero in the vicinity of the Bay's mouth. The contour pattern is linked to the system of currents in the Bay at the time, shown vectorially in figure 12. On examining the flow distribution of figure 12 it is apparent that, in the inner part of the Bay, the currents in the shallow near-shore regions on both sides are driven with the wind. A countercurrent against the wind along the deeper central axis must be driven by the pressure gradient associated with the wind

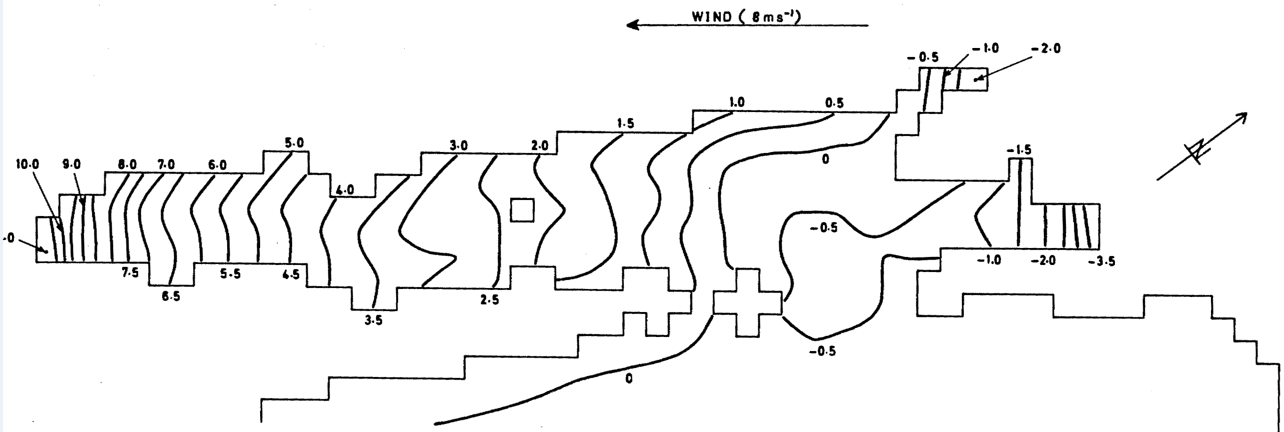


Figure 11. Surface elevations in Green Bay induced by wind field W_L, 40 h after the onset of the wind. Contours are marked in centimetres. Results from model 3.

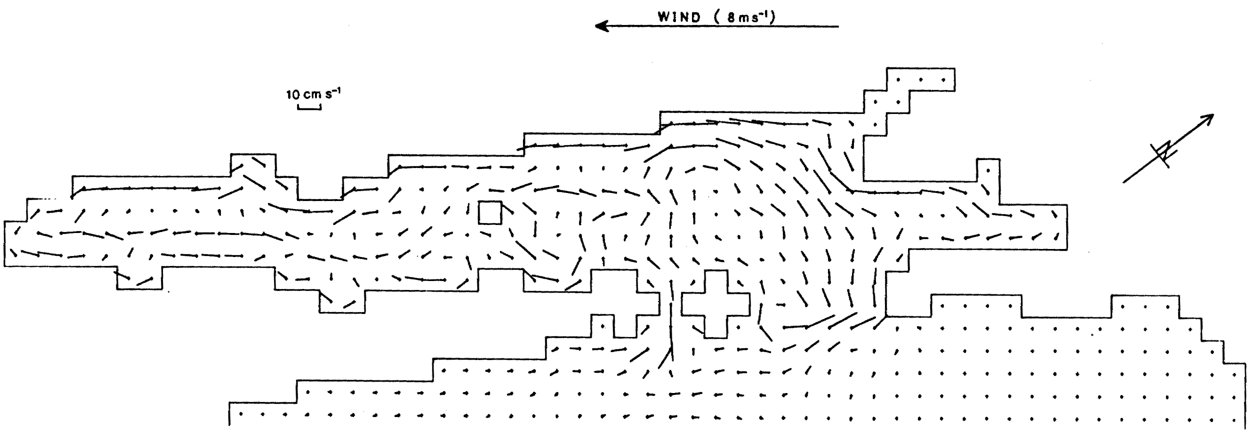


Figure 12. Current vectors in Green Bay induced by wind field W_L, 40 h after the onset of the wind. The origin of each vector is marked by a small cross. Results from model 3.

set-up. In the wider northern part of the Bay there is a large counterclockwise circulation with water flowing into the Bay from the Lake on the northern side of St Martin Island. Outward flow into the Lake occurs on both sides of Washington Island and in particular through Death's Door Passage where Plum Island is located. As reported by Mortimer (1979) and illustrated here in figure 13, there was indeed a steady out-going flow through that passage during a real northerly wind on 27–28 July 1969.

The model also displays smaller circulation patterns; these can be mainly associated with irregularities in depth topography and coastal configuration. Directly driven wind currents tend to develop in the shallower regions and countercurrents in the deeper areas. Notably, in the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

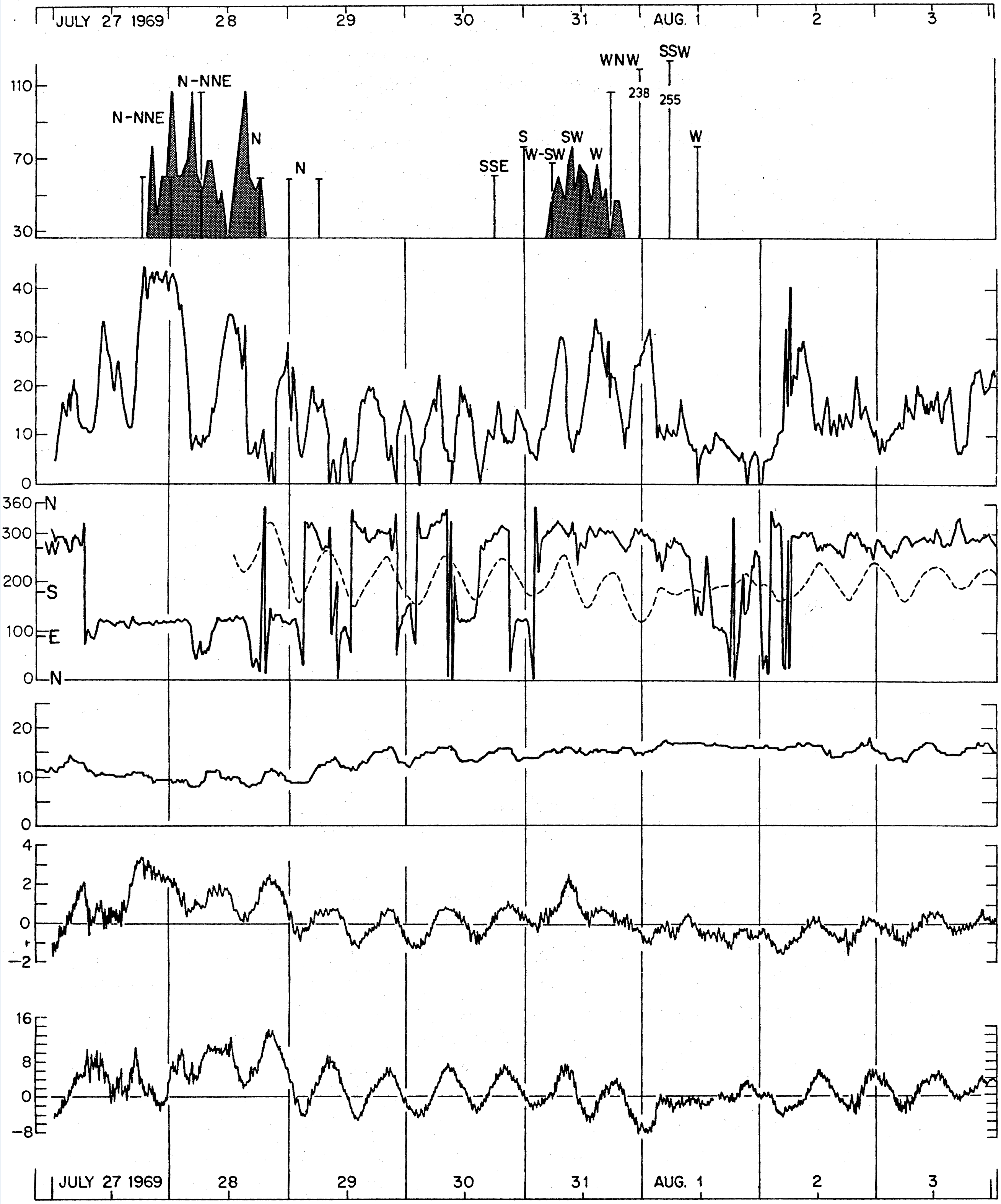


FIGURE 13. Current speed and direction, and also water temperature, at a midwater station (15 m depth, position indicated by the asterisk in figure 2*b*) in Death's Door Passage between Green Bay and Lake Michigan. Also shown is water level at nearby Plum Island and at Menominee (P and M, respectively, in figure 2) and wind speed (in excess of 5 m s^{-1}; squared) at Green Bay Airport and at Plum Island. The Menominee record is repeated, as a broken line, in the panel showing the direction of the current.

former case, the wind drives a current in its own direction along almost the entire length of the western coastline of the Bay. Regions of greatest horizontal shear, in which high horizontal turbulence may be expected, are encountered along the eastern shore in the inner part of the Bay and along the western shore to the northwest of centrally located Chambers Island. Many of the changes in inclination of the elevation contours in figure 11 may be explained in terms of changes in current direction, from one place to another, shown in figure 12. Thus, under the influence of the Earth's rotation, water level tends to slope up to the right looking in the direction of the current so that a change in the direction of the current with position will produce a corresponding change in surface slope and thence a change in contour inclination.

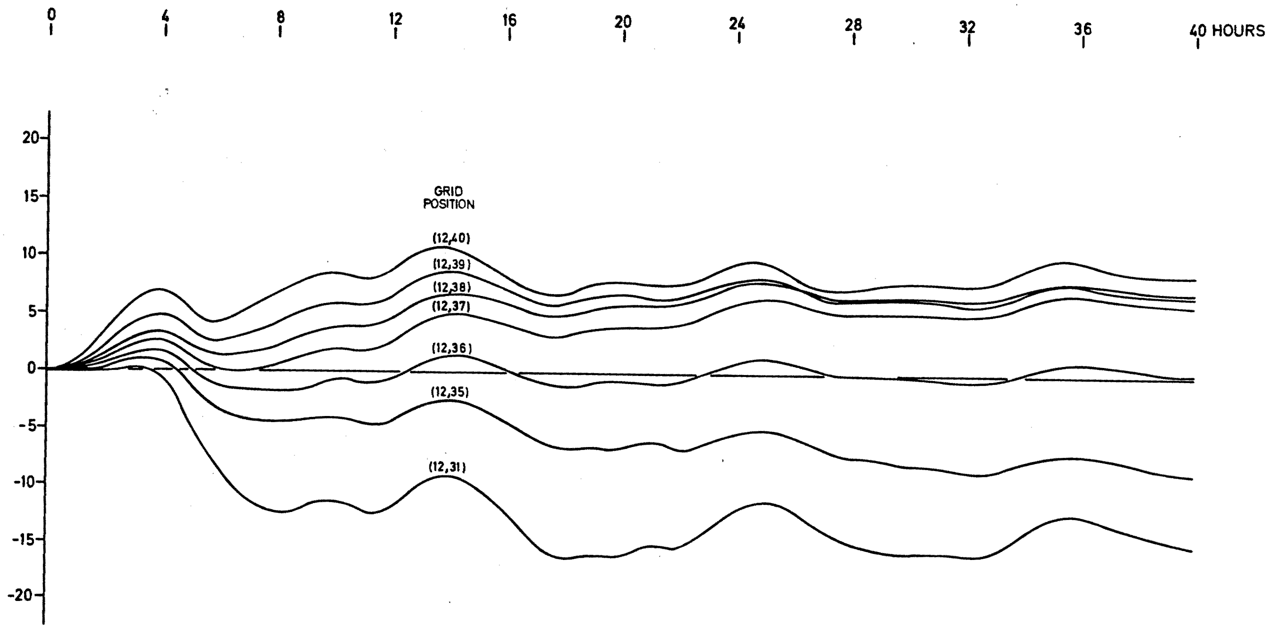


FIGURE 14. Time variations of v current directed normally into Green Bay across its mouth, induced by wind field W_L. Results from model 3. Grid points across the mouth, associated with these currents, are evident in figure 4.

From the computations, the circulation in the Bay (presented as in figure 12) was obtained at hourly intervals following the onset of the wind. The circulation pattern changes through time owing to the oscillatory motion, but after about 12 h settles down to essentially that shown in figure 12. Time variations of current directed normally across the mouth of the Bay are shown in figure 14. Manifestly, after an initial flow pulse into the Bay across the mouth, the currents quite quickly take up positive (inflow) values in the north and negative (outflow) values in the south in accordance with the vectors shown in figure 12. Time variations of longitudinal current at positions GB, $G_1, \ldots, G_8$ along the Bay's axis are plotted in figure 15. An initial flow towards the head under direct wind forcing occurs at all the positions; but subsequently the currents adjust to mean values shown in figure 12. The oscillatory motion along the Bay clearly contains evidence of the first and second Bay modes as do the elevation time plots, for the same positions, in figure 8.

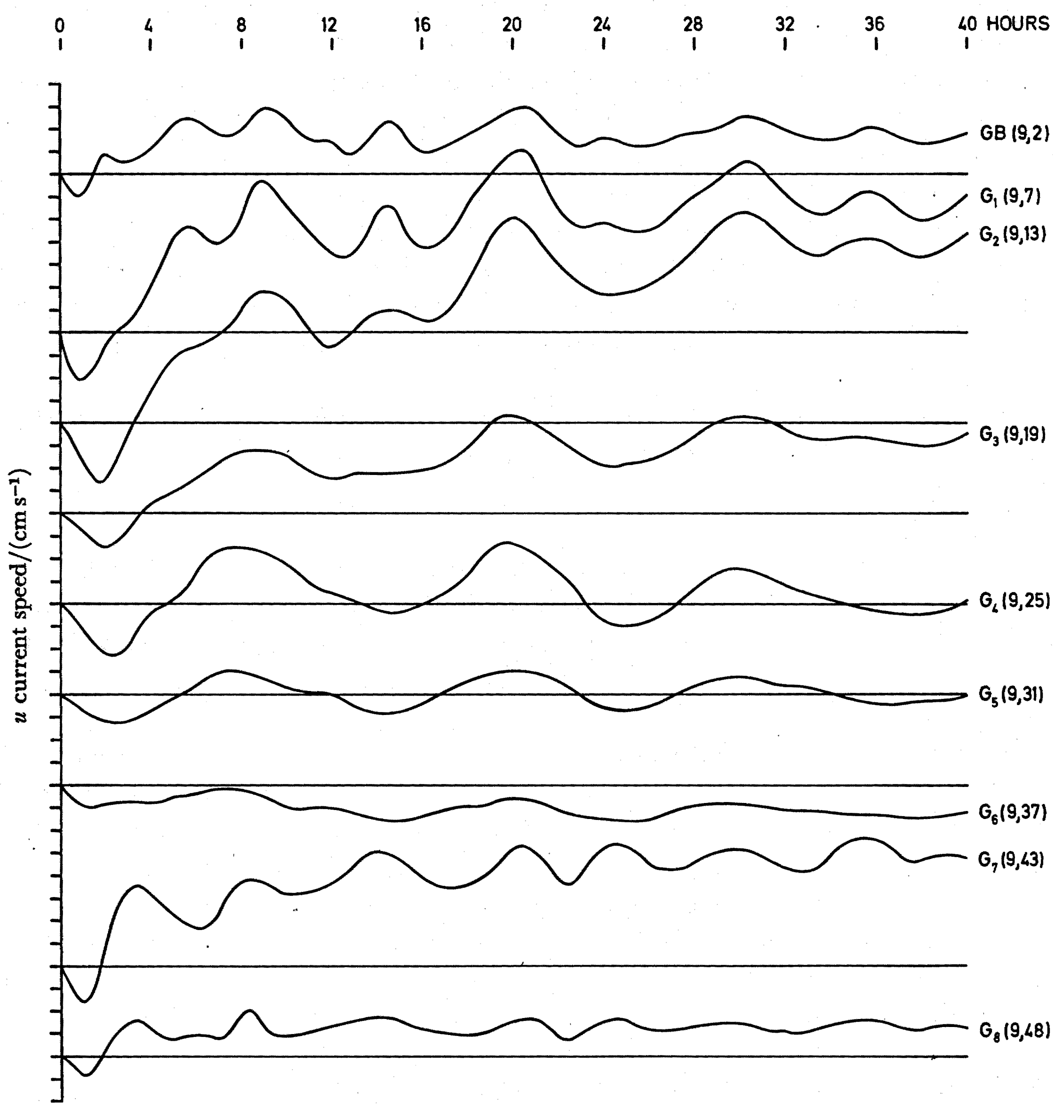


FIGURE 15. Longitudinal u currents in Green Bay due to wind field W_L. Time plots of current are shown in order corresponding to points GB, G_1–G_8 along the axis of the Bay (see figure 4). Results from model 3.

5. VERIFICATION OF MODEL 2 BY USING FIELD OBSERVATIONS

In an attempt to verify model 2, a field experiment was carried out during 1969 and 1970 in which temporary water level recorders were placed at Menominee and on Plum Island and St Martin Island at the mouth of the Bay. Also, current meters were deployed in the mouth channels near those islands. In addition, simultaneous records of water level were available from the permanently established gauge at Green Bay City. The water level observations at the mouth provided open-boundary conditions for the model, while those in the interior checked surface elevations derived from it. Hourly wind observations at Green Bay City Airport were assumed to give representative data for calculating the time-varying wind stress acting over the entire water surface of the Bay. Some justification for this assumption was provided by the similarity of winds observed at intervals of 3 h at Plum Island by the United States Coastguard.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

With use of model 2, continuous simulations of the motion in the Bay were done for two periods, one of 4 days (00 h 00 on 17 September to 00 h 00 on 21 September) and another of 8 days (09 h 00 on 7 October to 09 h 00 on 15 October), during 1969. Input to the model consisted of:

(*a*) A mean water level, defining the undisturbed surface level of the Bay, determined from records at Green Bay City and at Plum Island over the same 6 month interval.

(*b*) Half-hourly deviations from that mean of the actual surface elevations measured at Plum Island. These were applied successively through time across the mouth of the Bay (equation (6)). Between consecutive half-hourly specifications the imposed level was assumed to remain constant. The water levels measured at St Martin Island were found to be very similar to those at Plum Island and therefore it was judged unnecessary to allow for any positional variation in elevation across the mouth.

(*c*) Hourly values of wind speed and direction (unsmoothed) from Green Bay City Airport. Corresponding wind stress components were calculated from equations (11) and (12) and

$$F_s = -\tau\cos(38^\circ-\alpha), \quad G_s = -\tau\sin(38^\circ-\alpha),$$

where α is the wind angle in degrees, measured clockwise from the north, and 38° is the angle of inclination of the Bay's longitudinal axis, the x axis in figure 3, to the north. Hourly values of F_s, G_s thus obtained were applied successively through time over the water surface of the model (equation (5)). Between consecutive hourly specifications F_s, G_s were assumed to remain constant.

The simulation for each period was started from rest (equation (4)) and the model then computed the motion in the Bay through time under the action of the prescribed time-varying wind stress field and mouth elevation. Output of ζ, u, v was obtained at half-hourly intervals over the grid network of the model (figure 3). In particular, the variations of elevation ζ at Green Bay City, GB, and at Menominee, M, were determined for comparison with the recorded elevations at those locations. The predicted and the recorded elevations at GB are plotted together against time for the September period in figure 16*b* and for the October period in figure 17*b*. The input elevation at the Bay mouth is also shown plotted against time in these figures along with wind speed and wind angle (figures 16*a*, 17*a*) from the Green Bay City records. In figure 17*c* the predicted and recorded elevations at M for the October period are also compared.

Unfortunately, after the computations described above had been completed it was discovered that the current meters had malfunctioned after the middle of August 1969. Therefore no direct comparison between modelled and observed currents is possible; but information before the breakdown (see figure 13) will be referred to in later discussion.

First consider figure 16, covering the simulation of water levels at GB during the period 17–20 September 1969. The agreement between the predicted and recorded levels is fair during the 17th, deteriorates somewhat during the 18th, and becomes very good during the 19th and the 20th. Over the last two days a tidal oscillation is in evidence at the mouth of the Bay, and one may reasonably conclude that this oscillation is primarily responsible for the periodic-type response at GB on those days. The good agreement obtained between observation and theory in these circumstances suggests that the model responds correctly to excitation by mouth elevation. The disagreements between observation and theory on the 17th and on the 18th coincide with a situation in which there is little such mouth excitation. During that time the model response at GB consists of a set of fairly well ordered oscillations; these must originate, at least in part,

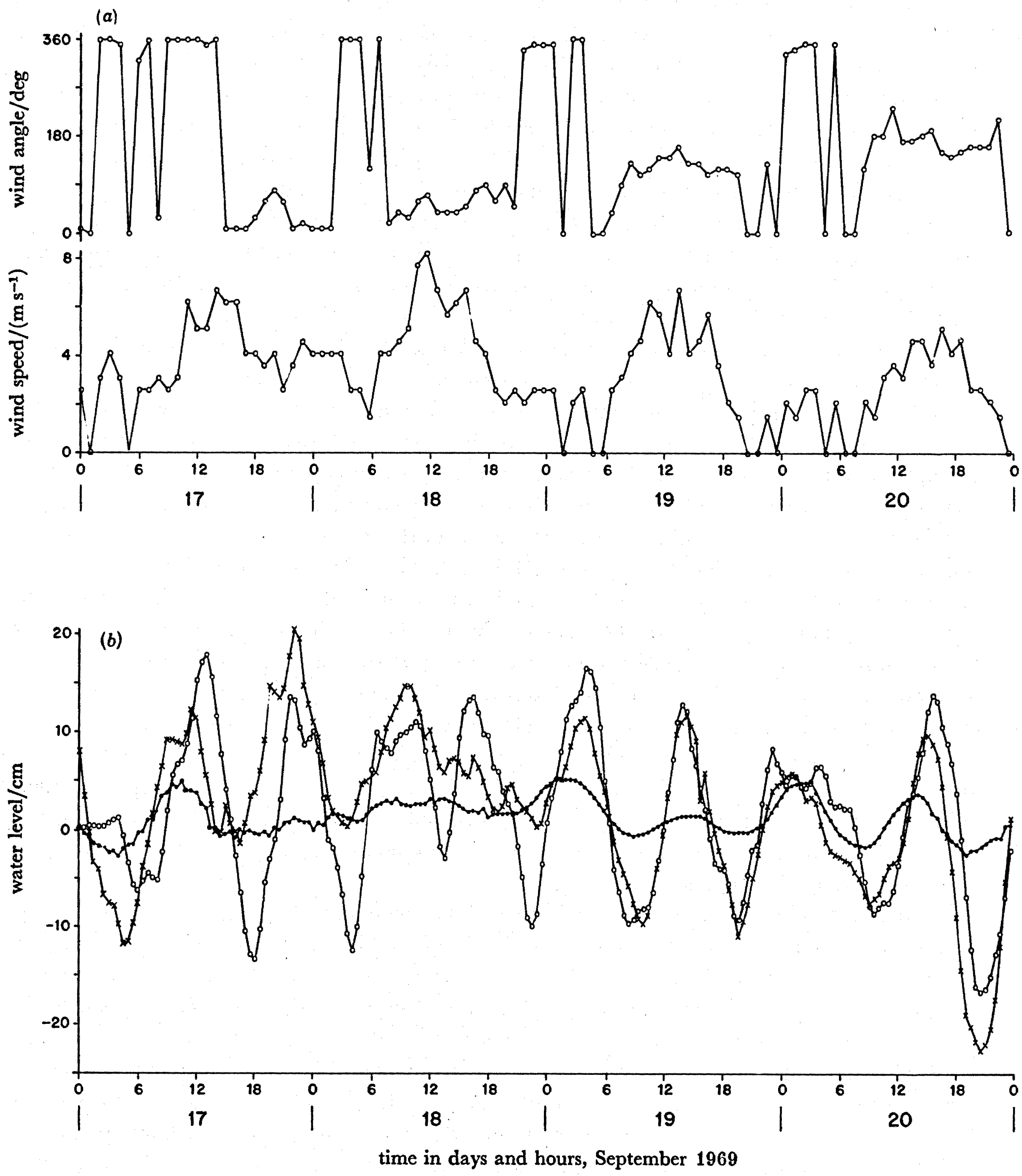


FIGURE 16. Numerically simulated (model 2) and observed water levels at GB (Green Bay City) during the period 00 h 00 on 17 September to 00 h 00 on 21 September 1969. (*a*) Wind speed and direction at Green Bay Airport. (*b*) Water levels. Key: ×, recorded water level at GB; o, water level at GB predicted by the model; •, recorded water level at Bay mouth, from Plum Island, used as model input.

from an elevation wave evident at the mouth on the 17th. However, the corresponding response at GB in reality, while displaying similar oscillations, is more irregular and exhibits a significant rise in mean water level scarcely reproduced by the model. We cannot give a completely satisfactory explanation for these discrepancies. They may be due to the use, in the model, of the fictitious initial condition of no motion, its influence propagating forward in time contaminating the true values. At some times the wind records (from the airport) used as input may be less representative of winds over the Bay than at others. Further, the wind peaks during the 17th and 18th, associated with winds from between north and northeast blowing down the Bay towards the head, might have induced Bay oscillations (as in §4) not adequately modelled because of an incorrect estimate of the forcing effect of winds from that direction. On the other hand, Bay oscillations might have been induced by current surges across the mouth with elevation near to zero there; the model, with an elevation open-boundary condition, could have missed these effects. The failure of the model to account for the observed rise in mean water level might be interpreted as an indication that the wind stresses applied to the model were too low.

Next consider figure 17*b*, in which the predicted and recorded water levels at GB for 7–15 October 1969 are compared. The winds during that period are generally stronger than those in the September period discussed above, reaching a maximum of 12 m s^{-1} on the 9th. They are westerly to southerly up to about the end of the 10th, then northerly to northeasterly until about midday on the 13th, then westerly to southerly again for the remainder of the time. By examining the water level variations it can be seen that, during the first 3 days, the calculated GB elevations are some 10 to 20 cm too high but are none the less adjusting to the observed oscillatory pattern. Subsequently, during the 10th, the 11th and the first 9 h of the 12th, there is very good agreement between observation and theory. Over this time span, the variations in level shown are such that a distinctive set of mouth oscillations would appear to be mainly responsible for a corresponding and equally distinctive set of GB oscillations. However, from 09 h 00 on the 12th to about 12 h 00 on the 13th, some large differences appear between the calculated and observed levels at GB. In essence, the observed levels swing up to two remarkably high successive peaks which are in no way accounted for by the model results. Subsequently, on the 14th and 15th, the calculated and observed levels come back into a close oscillatory correspondence, albeit the former are 10 to 15 cm higher than the latter.

The cause of the substantial differences between the model and observed levels at GB, during the 24 h starting at noon on 12 October, has not been established. Similar deviations are seen for that interval at Menominee (figure 17*c*). This behaviour cannot be attributed to any obvious change in wind speed or direction, although it should be noted that the direction was fairly steady (20–60° approximately from mouth to head of the Bay) during the interval in question and during the preceding day, i.e. a direction appropriate for excitation of the first Bay mode as described in §4. After the sudden reversal of wind direction, which occurred at about 08 h 00 on 13 October, the computed and observed levels at both GB and M come back into phase.

There is the possibility that, during the interval of disagreement, a rare combination of a strongly excited first Bay mode, the M_2 tide and the first Lake mode (motions discussed in §3) may have produced the observed result; but there is no explanation of why the model did not

FIGURE 17. Numerically simulated (model 2) and observed water levels during the period 09 h 00 on 7 October to 09 h 00 on 15 October 1969. Key as in figure 16. (*a*) Wind speed and direction at Green Bay Airport. (*b*) Water levels at Green Bay City. (*c*) Water levels at Menominee (×, recorded; o, predicted). The gaps in the record of observed levels were caused by faulty operation of the pen reversal mechanism in that range.

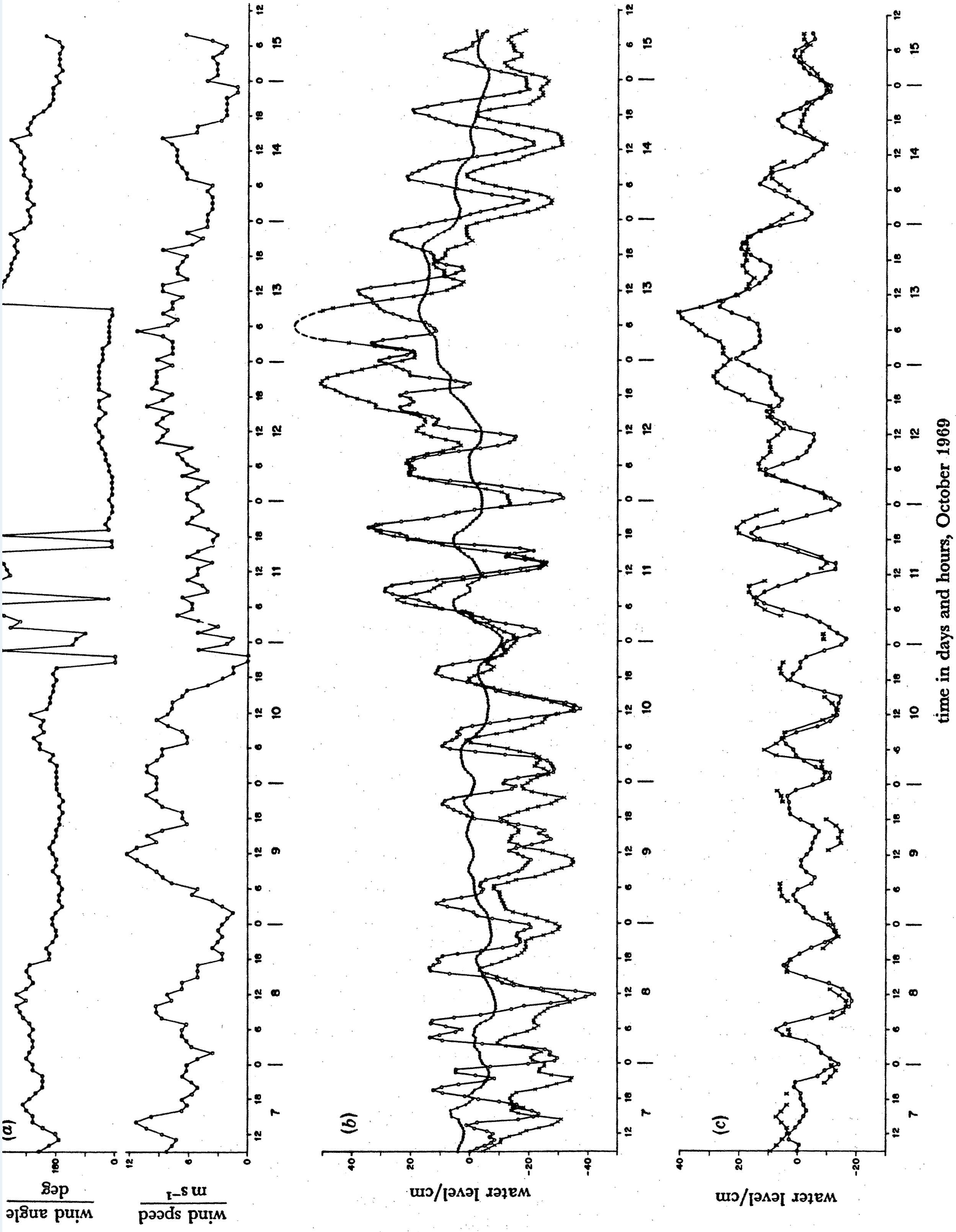


FIGURE 17. For description see facing page.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

simulate this effect, if it occurred. It appears to be significant that the interval of greatest discrepancy coincides with a large rise in mean level at the mouth, but with weak oscillations there. That rise is evidence of large motions in the Lake which may have affected the Bay in ways not encompassed by the model. This points to the necessity of including the whole Lake in the model in addition to improving the estimate of wind stress distribution. Evidence in favour of the idea of a conjuncture of modes comes from the mainly non-oscillatory form of the input water level at that time, conceivably indicating a cancellation of oscillatory forms at the mouth.

During 12 and 13 October, changes in mean water level clearly contributed substantially to the discrepancies between model and prototype. For present purposes, mean level is defined by a medial curve drawn between the successive crests and troughs of a water level trace. Thus, the observed mean water level at GB rises and falls through about 40 cm over the 2 days. However, the corresponding predicted level at GB only rises and falls through about 20 cm. At Menominee a similar behaviour is apparent with rises in observed and predicted mean level of some 20 and 10 cm respectively. Thus, in terms of mean variations, the model only partially reproduces the increases in level and the increase in longitudinal surface gradient in the inner part of the Bay. This suggests an underestimate of those wind stresses that are directed towards the head and force water into the inner region. Further, the flow into and then out of the whole Bay during the 2 day period may have been underestimated by the model.

Concerning the calculation of surface stress, we have to question whether the winds used are sufficiently representative of open-Bay conditions. Also the use of the drag coefficient given by (12) with comparatively light winds probably needs investigation. Some account should be taken of the state of the water surface when estimating drag force (Donelan 1978). Inflow and outflow through the mouth of the Bay are regulated in part by the elevation boundary condition there. When the input elevation becomes very small, a flow boundary condition might be more sensitively correlated to the Bay's motion and thereby give more accurate results. Therefore, measurements of current across the mouth are certainly desirable both to check the flow input and possibly to provide data for the formulation of an alternative open-boundary condition. For example, during the calm spell of 2–3 August 1969 (figure 13), oscillations of water level, as at Plum Island, were reduced in amplitude, but a relatively steady current averaging about 13 cm s^{-1} was maintained into the Bay through Death's Door Passage. This is in contrast to an in–out flow pulsation accompanied by larger and regular oscillations in level at Plum Island after the northerly wind impulses on 28 July. Use of the model to reproduce such different types of behaviour, involving flow as well as water level variations at the mouth, would contribute significantly to its further validation.

Reviewed overall, the model simulations of water level in figures 16 and 17 provide encouraging evidence that, in spite of some unresolved problems, the model performs reasonably well. Clearly, further testing against observational data is needed with a better specification of the surface stress condition and including a study of the use of a flow boundary condition at the mouth as an alternative to specifying elevation there. Simultaneous measurements of water level and current are required, with wind recording at one or two additional exposed locations along the length of the Bay.

Also in future work, given access to a more powerful computer, the model could be improved by the inclusion of the convective terms and the use of the finer grid (figure 2). To go even further, three-dimensional numerical models of the Bay's motion, taking account of summer stratification, may be envisaged. The extent to which that stratification influences the response of the Bay to wind pulses has still to be determined (Mortimer 1979, pp. 20–23).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

6. Concluding remarks

The development and use of numerical models of water motion in Green Bay have been described. The purpose has been to examine the response of the Bay to (*a*) long-wave disturbances originating in Lake Michigan and (*b*) wind over the Bay's water surface. The effects of these two forcing agents have been combined in a verification of computed water levels by using several days' observational data.

Perhaps the most important results of the work are contained in figures 5, 12, 16 and 17. Thus, figure 5 gives a frequency response curve (from model 2) which shows how a simple harmonic forcing oscillation of water level at the mouth of the Bay is magnified in amplitude and increased in phase in passing to the head. Realistic frequencies covered in the range of the response curve correspond to the first longitudinal mode of the Lake, the first longitudinal mode of the Bay, and the M_2 tide. Figure 12, on the other hand, shows the pattern of currents in the Bay (from model 3) associated with a wind blowing down the Bay towards the head. It will be interesting to see if future measurements of currents in the Bay reproduce this type of circulation. Finally, figures 16 and 17 compare observed water levels at Green Bay City and Menominee with predictions from model 2. Interest is in the discrepancies encountered in these comparisons, some of which might be due to inaccurate estimations of the wind stress field over the Bay and some to Lake motions not adequately considered. To validate the model, further field experiments involving measurements of both current and water level are required.

The dynamics of the Bay are inextricably connected with those of the Lake. The importance of the open-boundary condition in model 2 of the Bay demonstrates this, as also does the fact that it was found necessary in the study of wind effects in the Bay to extend model 2 into the northern part of the Lake to form model 3. To account properly for Bay–Lake interactions, the aim is to have a time-stepping model of the Bay and the *whole* of the Lake of similar design to, but of no coarser resolution than, the models employed here. Until such a model can be completed and run, some analytic studies of the excitation and superposition of different natural modes of oscillation in a simple bay–lake system, arranged to have a double resonance as in the Green Bay–Lake Michigan combination, might yield some insight into the conditions under which an optimum superposition of modes and the tide could arise to produce exceptionally large fluctuations in water level.

Most of the work described in this paper was done under the auspices of the University of Wisconsin Sea Grant Program during 1969 while N.S.H. was a Visiting Professor at the Center for Great Lakes Studies, University of Wisconsin–Milwaukee. The authors are indebted to Dr D. L. Cutchin at Milwaukee and Mr J. E. Jones at Bidston for help in the later stages of the project. Grateful acknowledgement is made to Mr D. F. Mraz for installing and maintaining the temporary water level recorders.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES